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Quantitative Finance FAQ

A peek into the world of quantitative finance.

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# What are the greeks?

Short Answer

The ‘greeks’ are the sensitivities of derivatives prices to underlyings, variables and parameters. They can be calculated by differentiating option values with respect to variables and/or parameters, either analytically, if you have a closed-form formula, or numerically.

Long Answer

**Delta**

The delta of an option or a portfolio of options is the sensitivity of the option or portfolio to the underlying. It is the rate of change of value with respect to the asset. Speculators take a view on the direction of some quantity such as the asset price and implement a strategy to take advantage of their view. If they own options then their exposure to the underlying is, to a first approximation, the same as if they own delta of the underlying.

Those who are not speculating on direction of the underlying will hedge by buying or selling the underlying, or another option, so that the portfolio delta is zero. By doing this they eliminate market risk.

Typically, the delta changes as stock price and time change, so to maintain a delta-neutral position the number of assets held requires continual readjustment by purchase or sale of the stock. This is called rehedging or rebalancing the portfolio and is an example of dynamic hedging.

Sometimes going short the stock for hedging purposes requires the borrowing of the stock in the first place. (You then sell what you have borrowed, buying it back later.) This can be costly, you may have to pay a repo rate, the equivalent of an interest rate, on the amount borrowed.

**Gamma**

The gamma of an option or a portfolio of options is the second derivative of the position with respect to the underlying. Since gamma is the sensitivity of the delta to the underlying it is a measure of by how much or how often a position must be rehedged to maintain a delta-neutral position. If there are costs associated with buying or selling stock, the bid–offer spread, for example, then the larger the gamma the larger the cost or friction caused by dynamic hedging.

Because costs can be large and because one wants to reduce exposure to model error it is natural to try to minimize the need to rebalance the portfolio too frequently. Since gamma is a measure of sensitivity of the hedge ratio to the movement in the underlying, the hedging requirement can be decreased by a gamma-neutral strategy. This means buying or selling more options, not just the underlying.

**Theta**

The theta is the rate of change of the option price with time. The theta is related to the option value, the delta, and the gamma by the Black–Scholes equation.

**Speed**

The speed of an option is the rate of change of the gamma with respect to the stock price. Traders use the gamma to estimate how much they will have to rehedge by if the stock moves. The stock moves by $1 so the delta changes by whatever the gamma is. But that’s only an approximation. The delta may change by more or less than this, especially if the stock moves by a larger amount, or the option is close to the strike and expiration. Hence the use of speed in a higher-order Taylor series expansion.

**Vega**

The vega, sometimes known as zeta or kappa, is a very important but confusing quantity. It is the sensitivity of the option price to volatility. This is completely different from the other greeks since it is a derivative with respect to a parameter and not a variable. This can be important. It is perfectly acceptable to consider sensitivity to a variable, which does vary, after all. However, it can be dangerous to measure sensitivity to something, such as volatility, which is a parameter and may, for example, have been assumed to be constant. That would be internally inconsistent.

As with gamma hedging, one can vega hedge to reduce sensitivity to the volatility. This is a major step towards eliminating some model risk, since it reduces dependence on a quantity that is not known very accurately. There is a downside to the measurement of vega. It is only meaningful for options having single-signed gamma everywhere. For example, it makes sense to measure vega for calls and puts but not binary calls and binary puts. The reason for this is that call and put values (and options with single-signed gamma) have values that are monotonic in the volatility: increase the volatility in a call and its value increases everywhere. Contracts with a gamma that changes sign may have a vega measured at zero because as we increase the volatility the price may rise somewhere and fall somewhere else. Such a contract is very exposed to volatility risk, but that risk is not measured by the vega.

**Rho**

Rho is the sensitivity of the option value to the interest rate used in the Black–Scholes formula. In practice one often uses a whole term structure of interest rates, meaning a time-dependent rate r(t). Rho would then be the sensitivity to the level of the rates assuming a parallel shift in rates at all times. Rho can also be sensitivity to dividend yield, or foreign interest rate in a foreign exchange option.

**Charm**

The charm is the sensitivity of delta to time. This is useful for seeing how your hedge position will change with time, for example, up until the next time you expect to hedge. This can be important near expiration.

**Colour**

The colour is the rate of change of gamma with time.

**Vanna**

The vanna is the sensitivity of delta to volatility. This is used when testing sensitivity of hedge ratios to volatility. It can be misleading at places when gamma is small.

**Vomma or Volga**

The Vomma or Volga is the second derivative of the option value with respect to volatility. Because of Jensen’s Inequality, if volatility is stochastic the Vomma/Volga measures convexity due to random volatility and so gives you an idea of how much to add (or subtract) from an option’s value.

# Why hedge?

Short Answer

‘Hedging’ in its broadest sense means the reduction of risk by exploiting relationships or correlation (or lack of correlation) between various risky investments. The purpose behind hedging is that it can lead to an improved risk/return. In the classical Modern Portfolio Theory framework, for example, it is usually possible to construct many portfolios having the same expected return but with different variance of returns (‘risk’). Clearly, if you have two portfolios with the same expected return the one with the lower risk is the better investment.

Long Answer

You buy a call option; it could go up or down in value depending on whether the underlying go up or down. So now sell some stock short. If you sell the right amount short then any rises or falls in the stock position will balance the falls or rises in the option, reducing risk.

To help to understand why one might hedge it is useful to look at the different types of hedging.

Probably the most important distinction between types of hedging is between model-independent and model-dependent hedging strategies.

**Model-independent hedging**: An example of such hedging is put–call parity. There is a simple relationship between calls and puts on an asset (when they are both European and with the same strikes and expiries), the underlying stock and a zero-coupon bond with the same maturity. This relationship is completely independent of how the underlying asset changes in value. Another example is spot-forward parity. In neither case do we have to specify the dynamics of the asset, not even its volatility, to find a possible hedge. Such model-independent hedges are few and far between.

**Model-dependent hedging**: Most sophisticated finance hedging strategies depend on a model for the underlying asset. The obvious example is the hedging used in the Black–Scholes analysis that leads to a whole theory for the value of derivatives. In pricing derivatives, we typically need to at least know the volatility of the underlying asset. If the model is wrong, then the option value and any hedging strategy could also be wrong.

**Delta hedging**

One of the building blocks of derivatives theory is delta hedging. This is the theoretically perfect elimination of all risk by using a very clever hedge between the option and its underlying. Delta hedging exploits the perfect correlation between the changes in the option value and the changes in the stock price. This is an example of ‘dynamic’ hedging; the hedge must be continually monitored and frequently adjusted by the sale or purchase of the underlying asset. Because of the frequent rehedging, any dynamic hedging strategy is going to result in losses due to transaction costs. In some markets this can be very important.

The ‘underlying’ in a delta-hedged portfolio could be a traded asset, a stock for example, or it could be another random quantity that determines a price such as a risk of default. If you have two instruments depending on the same risk of default, you can calculate the sensitivities, the deltas, of their prices to this quantity and then buy the two instruments in amounts inversely proportional to these deltas (one long, one short). This is also delta hedging.

If two underlyings are very highly correlated, you can use one as a proxy for the other for hedging purposes. You would then only be exposed to basis risk. Be careful with this because there may be times when the close relationship breaks down.

If you have many financial instruments that are uncorrelated with each other than you can construct a portfolio with much less risk than any one of the instruments individually. With such a large portfolio you can theoretically reduce risk to negligible levels. Although this isn’t strictly hedging it achieves the same goal.

**Gamma hedging**

To reduce the size of each rehedge and/or to increase the time between rehedges, and thus reduce costs, the technique of gamma hedging is often employed. A portfolio that is delta hedged is insensitive to movements in the underlying if those movements are quite small. There is a small error in this due to the convexity of the portfolio with respect to the underlying. Gamma hedging is a more accurate form of hedging that theoretically eliminates these second-order effects. Typically, one hedges one, exotic, say, contract with a vanilla contract and the underlying. The quantities of the vanilla and the underlying are chosen to make both the portfolio delta and the portfolio gamma instantaneously zero.

**Vega hedging**

The prices and hedging strategies are only as good as the model for the underlying. The key parameter that determines the value of a contract is the volatility of the underlying asset. Unfortunately, this is a very difficult parameter to measure. Nor is it usually a constant as assumed in the simple theories. Obviously, the value of a contract depends on this parameter, and so to ensure that a portfolio value is insensitive to this parameter we can vega hedge. This means that we hedge one option with both the underlying and another option in such a way that both the delta and the vega, the sensitivity of the portfolio value to volatility, are zero. This is often quite satisfactory in practice but is usually theoretically inconsistent; we should not use a constant volatility (basic Black–Scholes) model to calculate sensitivities to parameters that are assumed not to vary. The distinction between variables (underlying asset price and time) and parameters (volatility, dividend yield, interest rate) is extremely important here. It is justifiable to rely on sensitivities of prices to variables, but usually not sensitivity to parameters. To get around this problem it is possible to independently model volatility, etc., as variables themselves. In such a way it is possible to build up a consistent theory.

**Margin hedging**

Often what causes banks, and other institutions, to suffer during volatile markets is not the change in the paper value of their assets but the requirement to suddenly come up with a large amount of cash to cover an unexpected margin call. Examples where margin has caused significant damage are Metallgesellschaft and Long Term Capital Management. Writing options is very risky. The downside of buying an option is just the initial premium, the upside may be unlimited. The upside of writing an option is limited, but the downside could be huge. For this reason, to cover the risk of default in the event of an unfavourable outcome, the clearing houses that register and settle options insist on the deposit of a margin by the writers of options. Margin comes in two forms: the initial margin and the maintenance margin. The initial margin is the amount deposited at the initiation of the contract. The total amount held as margin must stay above a prescribed maintenance margin. If it ever falls below this level then more money (or equivalent in bonds, stocks, etc.) must be deposited. The amount of margin that must be deposited depends on the particular contract. A dramatic market move could result in a sudden large margin call that may be difficult to meet. To prevent this situation, it is possible to margin hedge. That is, set up a portfolio such that a margin calls on one part of the portfolio are balanced by refunds from other parts. Usually, over-the-counter contracts have no associated margin requirements and so won’t appear in the calculation.

# What is marketing to market and how does it affect risk management?

Short Answer

Marking to market means valuing an instrument at the price at which it is currently trading in the market. If you buy an option because you believe it is undervalued then you will not see any profit appear immediately, you will have wait until the market value moves into line with your own estimate. With an option this may not happen until expiration. When you hedge options, you must choose whether to use a delta based on the implied volatility or your own estimate of volatility. If you want to avoid fluctuations in your mark-to-market P&L you will hedge using the implied volatility, even though you may believe this volatility to be incorrect.

Long Answer

A stock is trading at $47, but you think it is seriously undervalued. You believe that the value should be $60. You buy the stock. How much do you tell people your little ‘portfolio’ is worth? $47 or $60? If you say $47 then you are marking to market, if you say $60 you are marking to (your) model. Obviously, this is open to serious abuse and so it is usual, and often a regulatory requirement, to quote the mark-to-market value. If you are right about the stock value, then the profit will be realized as the stock price rises.

If instruments are liquid, exchange traded, then marking to market is straightforward. You just need to know the most recent market-traded price. Of course, this doesn’t stop you also saying what you believe the value to be, or the profit you expect to make. After all, you presumably entered the trade because you thought you would make a gain.

Hedge funds will tell their investors their Net Asset Value based on the mark-to-market values of the liquid instruments in their portfolio. They may estimate future profit, although this is a bit of a hostage to fortune.

With futures and short options there are also margins to be paid, usually daily, to a clearing house as a safeguard against credit risk. So, if prices move against you, you may have to pay a maintenance margin. This will be based on the prevailing market values of the futures and short options. (There is no margin on long options positions because they are paid for up front, from which point the only way is up.) Marking to market of exchange-traded instruments is clearly very straightforward. But what about exotic or over the counter (OTC) contracts? These are not traded actively; they may be unique to you and your counterparty. These instruments must be marked to model. And this obviously raises the question of which model to use. Usually in this context the ‘model’ means the volatility, whether in equity markets, FX, or fixed income. So, the question about which model to use becomes a question about which volatility to use. With credit instruments the model often boils down to a number for risk of default.

Here are some possible ways of marking OTC contracts.

**The trader uses his own volatility**. Perhaps his best forecast going forward. This is very easy to abuse, it is very easy to rack up an imaginary profit this way. Whatever volatility is used it cannot be too far from the market’s implied volatilities on liquid options with the same underlying.

**Use prices obtained from brokers**. This has the advantage of being real, tradeable prices, and unprejudiced. The main drawback is that you can’t be forever calling brokers for prices with no intention of trading. They get very annoyed. And they won’t give you tickets to Wimbledon anymore.

**Use a volatility model that is calibrated to vanillas**. This has the advantage of giving prices that are consistent with the information in the market and are therefore arbitrage free. Although there is always the question of which volatility model to use, deterministic, stochastic, etc., so ‘arbitrage freeness’ is in the eye of the modeller. It can also be time consuming to have to crunch prices frequently.

One subtlety concerns the marking method and the hedging of derivatives. Take the simple case of a vanilla equity option bought because it is considered cheap. There are potentially three different volatilities here: implied volatility; forecast volatility; hedging volatility. In this situation the option, being exchange traded, would probably be marked to market using the implied volatility, but the ultimate profit will depend on the realized volatility (let’s be optimistic and assume it is as forecast) and how the option is hedged. Hedging using implied volatility in the delta formula theoretically eliminates the otherwise random fluctuations in the mark-to-market value of the hedged option portfolio, but at the cost of making the final profit path dependent, directly related to realized gamma along the stock’s path.

By marking to market or using a model-based marking that is as close to this as possible, your losses will be plain to see. If your theoretically profitable trade is doing badly, you will see your losses mounting up. You may be forced to close your position if the loss gets to be too large. Of course, you may have been right in the end, just a bit out in the timing. The loss could have reversed, but if you have closed out your position previously then tough. Having said that, human nature is such that people tend to hold onto losing positions too long on the assumption that they will recover, yet close out winning positions too early. Marking to market will therefore put some rationality back into your trading.

# What is calibration?

Short Answer

Calibration means choosing parameters in your model so that the theoretical prices for exchange-traded contracts output from your model match exactly, or as closely as possible, the market prices at an instant in time. In a sense it is the opposite of fitting parameters to historical time series. If you match prices exactly then you are eliminating arbitrage opportunities, and therefore it is popular.

Long Answer

You have your favourite interest rate model, but you don’t know how to decide what the parameters in the model should be. You realize that the bonds, swaps and swaptions markets are very liquid, and presumably very efficient. So, you choose your parameters in the model so that your model’s theoretical output for these simple instruments is the same as their market prices.

Almost all financial models have some parameter(s) that can’t be measured accurately. In the simplest non-trivial case, the Black–Scholes model, that parameter is volatility. If we can’t measure that parameter, how can we decide on its value? For if we don’t have an idea of its value then the model is useless.

Two ways spring to mind. One is to use historical data, the other is to use today’s price data.

Let’s see the first method in action. Examine, perhaps, equity data to try to estimate what volatility is. The problem with that is that it is necessarily backward looking, using data from the past. This might not be relevant to the future. Another problem with this is that it might give prices that are inconsistent with the market. For example, you are interested in buying a certain option. You think volatility is 27%, so you use that number to price the option, and the price you get is $15. However, the market price of that option is $19. Are you still interested in buying it? You can either decide that the option is incorrectly priced or that your volatility estimate is wrong.

The other method is to assume, effectively, that there is information in the market prices of traded instruments. In the above example we ask what volatility we must put into a formula to get the ‘correct’ price of $19. We then use that number to price other instruments. In this case we have calibrated our model to an instantaneous snapshot of the market at one moment in time, rather than to any information from the past.

Calibration is common in all markets but is usually more complicated than in the simple example above. Interest rate models may have dozens of parameters or even entire functions to be chosen by matching with the market.

Calibration can therefore often be time consuming. Calibration is an example of an inverse problem, in which we know the answer (the prices of simple contracts) and want to find the problem (the parameters). Inverse problems are notoriously difficult, for example being very sensitive to initial conditions.

Calibration can be misleading, since it suggests that your prices are correct. For example, if you calibrate a model to a set of vanilla contracts, and then calibrate a different model to the same set of vanillas, how do you know which model is better? Both correctly price vanillas today. But how will they perform tomorrow? Will you have to recalibrate? If you use the two different models to price an exotic contract, how do you know which price to use? How do you know which gives the better hedge ratios? How will you even know whether you have made money or lost it?

# What is bootstrapping using discount factors?

Short Answer

Bootstrapping means building up a forward interest-rate curve that is consistent with the market prices of common fixed-income instruments such as bonds and swaps. The resulting curve can then be used to value other instruments, such as bonds that are not traded.

Long Answer

You know the market prices of bonds with one, two three, five years to maturity. You are asked to value a four-year bond. How can you use the traded prices so that your four-year bond price is consistent?

Imagine that you live in a world where interest rates change in a completely deterministic way, no randomness at all. Interest rates may be low now, but rising in the future, for example. The spot interest rate is the interest you receive from one instant to the next. In this deterministic interest-rate world this spot rate can be written as a function of time, r(t). If you knew what this function was you would be able to value fixed-coupon bonds of all maturities by using the discount factor to present value a payment at time T to today, t.

Unfortunately, you are not told what this r function is. Instead, you only know, by looking at market prices of various fixed-income instruments, some constraints on this r function.

As a simple example, suppose you know that a zero-coupon bond, principal $100, maturing in one year, is worth $95 today. Suppose a similar two-year zero-coupon bond is worth $92.

This is hardly enough information to calculate the entire r(t) function, but it is like what we must deal with in practice. We have many bonds of different maturity, some without any coupons but most with, and very liquid swaps of various maturities. Each such instrument is a constraint on the r(t) function.

Bootstrapping is backing out a deterministic spot rate function, r(t), also called the (instantaneous) forward rate curve that is consistent with all these liquid instruments.

Note that usually only the simple ‘linear’ instruments are used for bootstrapping. Essentially this means bonds, but also includes swaps since they can be decomposed into a portfolio of bonds. Other contracts such as caps and floors contain an element of optionality and therefore require a stochastic model for interest rates. It would not make financial sense to assume a deterministic world for these instruments, just as you wouldn’t assume a deterministic stock price path for an equity option.

Because the forward rate curve is not uniquely determined by the finite set of constraints that we encounter in practice, we must impose some conditions on the function r(t).

Forward rates should be positive, or there will be arbitrage opportunities.

Forward rates should be continuous (although this is common-sense rather than because of any financial argument).

Perhaps the curve should also be smooth.

Even with these desirable characteristics the forward curve is not uniquely defined.

Finding the forward curve with these properties amounts to deciding on a way of interpolating ‘between the points,’ the ‘points’ meaning the constraints on the integrals of the r function. There have been many proposed interpolation techniques such as

linear in discount factors

linear in spot rates

linear in the logarithm of rates

piecewise linear continuous forwards

cubic splines

Bessel cubic spline

monotone-preserving cubic spline

quartic splines

Finally, the method should result in a forward rate function that is not too sensitive to the input data, the bond prices and swap rates, it must be fast to compute and must not be too local in the sense that if one input is changed it should only impact on the function nearby. And, of course, it should be emphasized that there is no ‘correct’ way to join the dots.

Because of the relative liquidity of the instruments, it is common to use deposit rates in the very short term, bonds and FRAs for the medium term and swaps for the longer end of the forward curve.

Because the bootstrapped forward curve is assumed to come from deterministic rates it is dangerous to use it to price instruments with convexity since such instruments require a model for randomness, as explained by Jensen’s Inequality.

Two other interpolation techniques are worth mentioning: first, that proposed by Jesse Jones and, second, the Epstein–Wilmott yield envelope.

The method proposed by Jesse Jones involves choosing the forward curve that satisfies all the constraints imposed by traded instruments but is, crucially, also not too far from the forward curve as found previously, the day before, say. The idea being simply that this will minimize changes in valuation for fixed-income instruments.

The Epstein–Wilmott model is nonlinear, posing constraints on the dynamics of the short rate. One of the outputs of this model is the Yield Envelope which gives no-arbitrage bounds on the forward curve.

# What is the volatility smile?

Short Answer

Volatility smile is the phrase used to describe how the implied volatilities of options vary with their strikes. A smile means that out-of-the-money puts and out-of-the-money calls both have higher implied volatilities than at-the-money options. Other shapes are possible as well. A slope in the curve is called a skew. So, a negative skew would be a download-sloping graph of implied volatility versus strike.

Long Answer

Let us begin with how to calculate the implied volatilities. Start with the prices of traded vanilla options, usually the mid-price between bid and offer, and all other parameters needed in the Black–Scholes formula, such as strikes, expirations, interest rates, dividends, except for volatilities. Now ask the question: What volatility must be used for each option series so that the theoretical Black–Scholes price and the market price are the same?

Although we have the Black–Scholes formula for option values as a function of volatility, there is no formula for the implied volatility as a function of option value, it must be calculated using some bisection, Newton–Raphson, or other numerical technique for finding zeros of a function. Now plot these implied volatilities against strike, one curve per expiration. That is the implied volatility smile. If you plot implied volatility against both strike and expiration as a three-dimensional plot, then that is the implied volatility surface. Often you will find that the smile is quite flat for long-dated options but getting steeper for short-dated options.

Since the Black–Scholes formula assume constant volatility (or with a minor change, time-dependent volatility) you might expect a flat implied volatility plot. This does not appear to be the case from real option-price data. How can we explain this? Here are some questions to ask.

Is volatility constant?

Are the Black–Scholes formula, correct?

Do option traders use the Black–Scholes formula?

Volatility does not appear to be constant. By this we mean that actual volatility is not constant, actual volatility being the amount of randomness in a stock’s return. Actual volatility is something you can try to measure from a stock price time series and would exist even if options didn’t exist. Although it is easy to say with confidence that actual volatility is not constant, it is altogether much harder to estimate the future behaviour of volatility. So that might explain why implied volatility is not constant, and people believe that volatility is not constant.

If volatility is not constant, then the Black–Scholes formula are not correct. (Again, there is the small caveat that the Black–Scholes formula can work if volatility is a known deterministic function of time. But I think we can also confidently dismiss this idea as well.)

Despite this, option traders do still use the Black–Scholes formula for vanilla options. Of all the models that have been invented, the Black–Scholes model is still the most popular for vanilla contracts. It is simple and easy to use, it has very few parameters, it is very robust. Its drawbacks are quite well understood. But very often, instead of using models without some of the Black–Scholes’ drawbacks, people ‘adapt’ Black–Scholes to accommodate those problems. For example, when a stock falls dramatically we often see a temporary increase in its volatility. How can that be squeezed into the Black–Scholes framework? Easy, just bump up the implied volatilities for option with lower strikes. A low strike put option will be out of the money until the stock falls, at which point it may be at the money, and at the same time volatility might rise. So, bump up the volatility of all the out-of-the-money puts. This deviation from the flat-volatility Black–Scholes world tends to get more pronounced closer to expiration.

A more general explanation for the volatility smile is that it incorporates the kurtosis seen in stock returns. Stock returns are not normal, stock prices are not lognormal. Both have fatter tails than you would expect from normally distributed returns. We know that, theoretically, the value of an option is the present value of the expected payoff under a risk-neutral random walk. If that risk-neutral probability density function has fat tails, then you would expect option prices to be higher than Black–Scholes for very low and high strikes. Hence higher implied volatilities, and the smile.

Another school of thought is that the volatility smile and skew exist because of supply and demand. Option prices come less from an analysis of probability of tail events than from simple agreement between a buyer and a seller. Out-of-the-money puts are a cheap way of buying protection against a crash. But any form of insurance is expensive; after all, those selling the insurance also want to make a profit. Thus out-of-the-money puts are relatively over-priced. This explains high implied volatility for low strikes. At the other end, many people owning stock will write out-of-the-money call options (so-called covered call writing) to take in some premium, perhaps when markets are moving sideways. There will therefore be an over-supply of out-of-the-money calls, pushing the prices down. Net result, a negative skew. Although the simple supply/demand explanation is popular among traders it does not sit comfortably with quants because it does suggest that options are not correctly priced and that there may be arbitrage opportunities.

While on the topic of arbitrage, it is worth mentioning that there are constraints on the skew and the smile that come from examining simple option portfolios. For example, rather obviously, the higher the strike of a call option, the lower its price. Otherwise, you could make money rather easily by buying the low strike call and selling the higher strike call. This imposes a constraint on the skew. Similarly, a butterfly spread must have a positive value since the payoff can never be negative. This imposes a constraint on the curvature of the smile. Both constraints are model independent.

There are many ways to build the volatility-smile effect into an option-pricing model, and still have no arbitrage. The most popular are, in order of complexity, as follows

Deterministic volatility surface

Stochastic volatility

Jump diffusion

The deterministic volatility surface is the idea that volatility is not constant, or even only a function of time, but a known function of stock price and time, σ (S, t). Here the word ‘known’ is a bit misleading. What we really know are the market prices of vanilla options, a snapshot at one instant in time. We must now figure out the correct function σ (S, t) such that the theoretical value of our options matches the market prices. This is mathematically an inverse problem, essentially find the parameter, volatility, knowing some solutions, market prices. This model may capture the volatility surface exactly at an instant in time, but it does a very poor job of capturing the dynamics, that is, how the data change with time.

Stochastic volatility models have two sources of randomness, the stock return, and the volatility. One of the parameters in these models is the correlation between the two sources of randomness. This correlation is typically negative so that a fall in the stock price is often accompanied by a rise in volatility. This results in a negative skew for implied volatility. Unfortunately, this negative skew is not usually as pronounced as the real market skew. These models can also explain the smile. As a rule, one pays for convexity. We see this in the simple Black–Scholes world where we pay for gamma. In the stochastic volatility world, we can look at the second derivative of option value with respect to volatility, and if it is positive, we would expect to have to pay for this convexity – that is, option values will be relatively higher wherever this quantity is largest.

Stochastic volatility models have greater potential for capturing dynamics, but the problem, as always, is knowing which stochastic volatility model to choose and how to find its parameters. When calibrated to market prices you will still usually find that supposed constant parameters in your model keep changing. This is often the case with calibrated models and suggests that the model is still not correct, even though its complexity seems to be very promising.

Jump-diffusion models allow the stock (and even the volatility) to be discontinuous. Such models contain so many parameters that calibration can be instantaneously more accurate (if not necessarily stable through time).

# What is Monte Carlo simulation?

Short Answer

Monte Carlo simulations are a way of solving probabilistic problems by numerically ‘imagining’ many possible scenarios or games to calculate statistical properties such as expectations, variances, or probabilities of certain outcomes. In finance we use such simulations to represent the future behaviour of equities, exchange rates, interest rates, etc., to either study the possible future performance of a portfolio or to price derivatives.

Long Answer

We hold a complex portfolio of investments; we would like to know the probability of losing money over the next year since our bonus depends on our making a profit. We can estimate this probability by simulating how the individual components in our portfolio might evolve over the next year. This requires us to have a model for the random behaviour of each of the assets, including the relationship or correlation between them, if any. Some problems which are completely deterministic can also be solved numerically by running simulations, most famously finding a value for π.

It is clear enough that probabilistic problems can be solved by simulations. What is the probability of tossing heads with a coin, just toss the coin often enough and you will find the answer. More on this and its relevance to finance shortly. But many deterministic problems can also be solved this way, provided you can find a probabilistic equivalent of the deterministic problem. A famous example of this is Buffon’s needle, a problem and solution dating back to 1777. Draw parallel lines on a table one inch apart. Drop a needle, also one inch long, onto this table. Simple trigonometry will show you that the probability of the needle touching one of the lines is 2/π. So, conduct many such experiments to get an approximation to π. Unfortunately, because of the probabilistic nature of this method you will have to drop the needle many billions of times to find π accurate to half a dozen decimal places.

There can also be a relationship between certain types of differential equation and probabilistic methods. Stanislaw Ulam, inspired by a card game, invented this technique while working on the Manhattan Project towards the development of nuclear weapons. The name ‘Monte Carlo’ was given to this idea by his colleague Nicholas Metropolis.

Monte Carlo simulations are used in financial problems for solving two types of problems:

Exploring the statistical properties of a portfolio of investments or cashflows to determine quantities such as expected returns, risk, possible downsides, probabilities of making certain profits or losses, etc.

Finding the value of derivatives by exploiting the theoretical relationship between option values and expected payoff under a risk-neutral random walk.

**Exploring portfolio statistics**

The most successful quantitative models represent investments as random walks. There is a whole mathematical theory behind these models, but to appreciate the role they play in portfolio analysis you just need to understand three simple concepts.

First, you need an algorithm for how the most basic investments evolve randomly. In equities this is often the lognormal random walk. (If you know about the real/risk-neutral distinction then you should know that you will be using the real random walk here.) This can be represented on a spreadsheet or in code as how a stock price changes from one period to the next by adding on a random return. In the fixed-income world you may be using the BGM model to model how interest rates of various maturities evolve. In credit you may have a model that models the random bankruptcy of a company. If you have more than one such investment that you must model, then you will also need to represent any interrelationships between them. This is often achieved by using correlations.

Once you can perform such simulations of the basic investments then you need to have models for more complicated contracts that depend on them, these are the options/derivatives/contingent claims. For this you need some theory, derivatives theory. This the second concept you must understand.

Finally, you will be able to simulate many thousands, or more, future scenarios for your portfolio and use the results to examine the statistics of this portfolio. This is, for example, how classical Value at Risk can be estimated, among other things.

**Pricing derivatives**

We know from the results of risk-neutral pricing that in the popular derivatives theories the value of an option can be calculated as the present value of the expected payoff under a risk-neutral random walk. And calculating expectations for a single contract is just a simple example of the above-mentioned portfolio analysis, but just for a single option and using the risk-neutral instead of the real random walk. Even though the pricing models can often be written as deterministic partial differential equations they can be solved in a probabilistic way, just as Stanislaw Ulam noted for other, non-financial, problems. This pricing methodology for derivatives was first proposed by the actuarially trained Phelim Boyle in 1977.

Whether you use Monte Carlo for probabilistic or deterministic problems the method is usually quite simple to implement in basic form and so is extremely popular in practice.

# How is risk defined in mathematical terms?

Short Answer

In layman’s terms, risk is the possibility of harm or loss. In finance it refers to the possibility of a monetary loss associated with investments.

Long Answer

The most common measure of risk is simply standard deviation of portfolio returns. The higher this is, the more randomness in a portfolio, and this is seen as a bad thing.

Financial risk comes in many forms:

**Market risk**: The possibility of loss due to movements in the market, either as a whole or specific investments

**Credit risk**: The possibility of loss due to default on a financial obligation

**Model risk**: The possibility of loss due to errors in mathematical models, often models of derivatives. Since these models contain parameters, such as volatility, we can also speak of parameter risk, volatility risk, etc.

**Operational risk**: The possibility of loss due to people, procedures, or systems. This includes human error and fraud

**Legal risk**: The possibility of loss due to legal action or the meaning of legal contracts

Before looking at the mathematics of risk we should understand the difference between risk, randomness, and uncertainty, all of which are important.

When measuring risk, we often use probabilistic concepts. But this requires having a distribution for the randomness in investments, a probability density function, for example. With enough data or a decent enough model, we may have a good idea about the distribution of returns. However, without the data, or when embarking into unknown territory we may be completely in the dark as to probabilities. This is especially true when looking at scenarios which are incredibly rare or have never even happened before. For example, we may have a good idea of the results of an alien invasion, after all, many scenarios have been explored in the movies, but what is the probability of this happening? When you do not know the probabilities then you have what Knight (1921) termed ‘uncertainty.’

We can categorize these issues, following Knight, as follows.

1. For ‘risk’ the probabilities that specified events will occur in the future are measurable and known, i.e., there is randomness but with a known probability distribution. This can be further divided.

(a) a priori risk, such as the outcome of the roll of a fair die

(b) estimable risk, where the probabilities can be estimated through statistical analysis of the past, for example, the probability of a one-day fall of 10% in the S&P index

2. With ‘uncertainty’ the probabilities of future events cannot be estimated or calculated.

In finance we tend to concentrate on risk with probabilities we estimate, we then have all the tools of statistics and probability for quantifying various aspects of that risk. In some financial models we do attempt to address the uncertain. For example, the uncertain volatility work of Avellaneda et al. (1995). Here volatility is uncertain, is allowed to lie within a specified range, but the probability of volatility having any value is not given. Instead of working with probabilities we now work with worst-case scenarios. Uncertainty is therefore more associated with the idea of stress-testing portfolios.

CrashMetrics is another example of worst-case scenarios and uncertainty.

A starting point for a mathematical definition of risk is simply as standard deviation. This is sensible because of the results of the Central Limit Theorem (CLT), that if you add up many investments what matters as far as the statistical properties of the portfolio are just the expected return and the standard deviation of individual investments, and the resulting portfolio returns are normally distributed. The normal distribution being symmetrical about the mean, the potential downside can be measured in terms of the standard deviation.

However, this is only meaningful if the conditions for the CLT are satisfied. For example, if we only have a small number of investments, or if the investments are correlated, or if they don’t have finite variance, then standard deviation may not be relevant.

Another mathematical definition of risk is semi variance, in which only downside deviations are used in the calculation. This definition is used in the Sortino performance measure.

Artzner et al. (1997) proposed a set of properties that a measure of risk should satisfy for it to be sensible. Such risk measures are called coherent.

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